Four Types of Improved Variable-Reduction Methods for Solving Asymmetric EFG-Type Saddle-Point Problem

Atsushi KAMITANI and Teruou TAKAYAMA

Graduate School of Science and Engineering, Yamagata University,
Yonezawa, Yamagata 992-8510, Japan

Abstract

Discretization of a boundary-value problem with the eXtended Element-Free Galerkin (X-EFG) method yields an asymmetric EFG-type Saddle-Point (EFG-SP) problem that is difficult to solve numerically. As high-performance solvers for the problem, four types of the Asymmetric-version improved Variable-Reduction Methods (AiVRMs) are formulated. A numerical code is developed for solving asymmetric EFG-SP problems with four types of AiVRMs and, by means of the code, performances of the four methods are investigated numerically.

1 Introduction

As a meshless approach for solving a boundary-value problem, the eXtended Element-Free Galerkin (X-EFG) method [1] was developed. If the X-EFG method is applied to a boundary-value problem of an elliptic partial differential equation, we obtain the following linear system [1–3]:

$$\begin{bmatrix} B & C \\ D^T & O \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix}, \tag{1}$$

where $B \in \mathbb{R}^{N\times N}$ and $C, D \in \mathbb{R}^{N\times K}$ are given matrices. Also, $\boldsymbol{c} \in \mathbb{R}^N$ and $\boldsymbol{d} \in \mathbb{R}^K$ are both given vectors, whereas $\boldsymbol{u} \in \mathbb{R}^N$ and $\boldsymbol{\lambda} \in \mathbb{R}^K$ are both unknown vectors. Here, N and K are two natural numbers satisfying $K < N$. If submatrices, B, C and D, fulfill the 6 conditions described in [3], (1) is called an asymmetric EFG-type Saddle-Point (EFG-SP) problem. As is well known, an asymmetric EFG-SP problem is difficult to solve numerically [4].

The purpose of the present study is to develop solvers for asymmetric EFG-SP problems and to evaluate their performance numerically.

2 Elimination of Lagrange Multiplier

In our earlier work [5], a high-performance solver was developed for symmetric EFG-SP problems and it is called the improved Variable-Reduction Method (iVRM). The basic idea of iVRM is to eliminate a vector corresponding to the Lagrange multiplier from a symmetric EFG-SP problem. In order to develop the Asymmetric-version iVRM (AiVRM), let us eliminate the vector $\boldsymbol{\lambda}$ from (1). To this end, we assume that $\mathbb{R}^N = \mathrm{Im}\, C \oplus W$. Moreover, we assume that two matrices, $\mathcal{U}_{\boldsymbol{\lambda}}$ and $\mathcal{F}_{\mathrm{B}}$, satisfy the following three conditions: i) $\mathcal{U}_{\boldsymbol{\lambda}}$ is a projection matrix along $\mathrm{Im}\, C$ onto W. ii) There exists an $N \times K$ matrix G with a full-column rank such that $\mathcal{F}_{\mathrm{B}} = GD^T$. iii) $\mathrm{Im}\, \mathcal{U}_{\boldsymbol{\lambda}} \cap \mathrm{Im}\, \mathcal{F}_{\mathrm{B}} = \{\boldsymbol{0}\}$.

By using $\mathcal{U}_{\boldsymbol{\lambda}}$ and $\mathcal{F}_{\mathrm{B}}$, we can eliminate $\boldsymbol{\lambda}$ from (1). As a result, we have

$$B^{\dagger}\boldsymbol{u} = \boldsymbol{c}^{\dagger}, \tag{2}$$

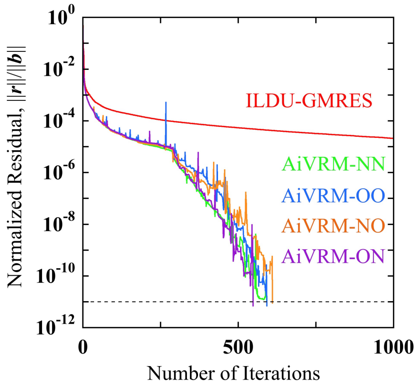

Figure 1: Residual histories of ILDU-GMRES and four types of AiVRMs for N = 1,050,625.

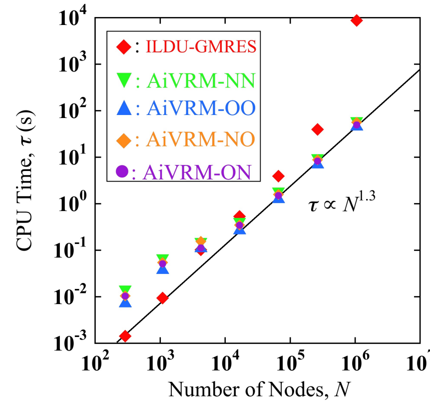

Figure 2: Dependence of the CPU time τ on the number N of nodes.

where $B^\dagger \equiv \mathcal{U}_{\boldsymbol{\lambda}}\, B\, \mathcal{U}_{\rm B} + \mathcal{F}_{\rm B}$, $\boldsymbol{c}^\dagger \equiv \mathcal{U}_{\boldsymbol{\lambda}}\,(\boldsymbol{c} - BG\boldsymbol{d}) + G\boldsymbol{d}$ and $\mathcal{U}_{\rm B} = I - \mathcal{F}_{\rm B}$. Here, I denotes an identity matrix. Throughout the present study, the numerical method for solving (2) is called AiVRM.

3 Asymmetric-version iVRM

Let us search for matrices satisfying the conditions, i), ii) and iii). As candidates for such matrices, the following six projection matrices are introduced: $F \equiv C(D^TC)^{-1}D^T$, $U \equiv I - F$, $F_C \equiv C(C^TC)^{-1}C^T$, $U_C \equiv I - F_C$, $F_D \equiv D(D^TD)^{-1}D^T$ and $U_D \equiv I - F_D$. After a straightforward algebra, we can show that the following four combinations of $(\mathcal{U}_{\boldsymbol{\lambda}}, \mathcal{F}_{\rm B})$ fulfill the conditions: $(\mathcal{U}_{\boldsymbol{\lambda}}, \mathcal{F}_{\rm B}) = (U, F)$, (U, F_D), (U_C, F) and (U_C, F_D). Throughout the present study, AiVRMs with $(\mathcal{U}_{\boldsymbol{\lambda}}, \mathcal{F}_{\rm B}) = (U, F)$, (U, F_D), (U_C, F) and (U_C, F_D) are called AiVRM-NN, AiVRM-NO, AiVRM-ON and AiVRM-OO, respectively.

In solving (2) with AiVRMs, multiplications of projection matrices and vectors are calculated. The matrix-vector multiplications require the solution of a linear system with K unknowns. Such a linear system is called an inner linear system, whereas (2) is called an outer linear system. Moreover, Krylov subspace methods for solving inner and outer linear systems are called inner and outer solvers, respectively. In the actual calculation, BiCGSTAB is always adopted as an outer solver in four types of AiVRMs. On the other hand, in solving inner linear systems with symmetric and asymmetric coefficient matrices, the conjugate gradient method and BiCGSTAB are employed as inner solvers, respectively.

An asymmetric EFG-SP problem originating from a 2D Poisson problem is solved with ILDU-GMRES and four types of AiVRMs. The residual histories and the dependence of the CPU time on the number N of nodes are depicted in Figures 1 and 2, respectively.

References

[1] A. Kamitani, T. Takayama, T. Itoh and H. Nakamura, *Plasma Fusion Res.* **6** (2011) Art. No. 2401074.

[2] T. Itoh, A. Saitoh, S. Ikuno and A. Kamitani, *J. Adv. Simulat. Sci. Eng.* **3** (2017) 188-205.

[3] A. Kamitani, T. Takayama and H. Nakamura, *Plasma Fusion Res.* **18** (2023) Art. No. 2403039.

[4] M. Benzi, G. H. Golub and J. Liesen, *Acta Numer.* **14** (2005) 1-137.

[5] A. Kamitani, T. Takayama and H. Nakamura, *Plasma Fusion Res.* **17** (2022) Art. No. 2403014.